MA 1115 **Exam #** 2

May 17**th,** 2007 **Name** ______________________________

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Show all necessary work in each problem to receive credit.

1. (20 points) for the curve $r(t) = \langle \sin t - t\cos t, \cos t + t \sin t, \frac{t^2}{2}\rangle$, find the unit normal vectors $T(t)$ and $N(t)$ and the curvature $(t > 0)$.

Solution: First note that $r'(t) = \langle \cos t - \cos t + t \sin t, -\sin t + \sin t + t\cos t, t\rangle = \langle t \sin t, t\cos t, t\rangle$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{\langle t\sin t, t\cos t, t\rangle}{\sqrt{t^2\sin^2 t + t^2\cos^2 t + t^2}} = \frac{\langle t\sin t, t\cos t, t\rangle}{t\sqrt{2}} = \frac{\langle \sin t, \cos t, 1\rangle}{\sqrt{2}}$$

$$T'(t) = \frac{\langle \cos t, -\sin t, 0\rangle}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\langle \frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\rangle}{\sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2} + 0}} = \frac{\langle \frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\rangle}{\frac{1}{\sqrt{2}}} = \langle \cos t, -\sin t, 0\rangle$$

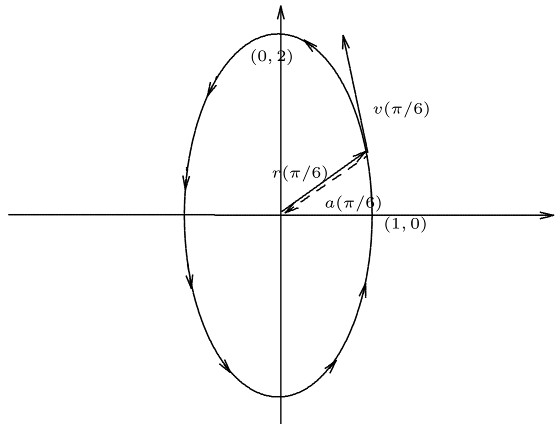
curvature: $\kappa = \dfrac{|T'(t)|}{|r'(t)|} = \dfrac{1/\sqrt{2}}{t\sqrt{2}} = \dfrac{1}{2t}, t > 0$

2. (20 points) Find the velocity, acceleration and speed of a particle with the position function $r(t) = \langle \cos t, 2\sin t\rangle$, $t = \pi/6$. Sketch the path of the particle and draw the velocity and acceleration vectors at the specified value.

Solution: $r(t) = \langle \cos t, 2\sin t\rangle$, with $r(\pi/6) = \langle \sqrt{3}/2, 1\rangle$
velocity: $v(t) = \langle -\sin t, 2\cos t\rangle$, with $v(\pi/6) = \langle -1/2, \sqrt{3}\rangle$

speed: $|v(\pi/6)| = \sqrt{\sin^2 \pi/6 + 4\cos^2 \pi/6} = \sqrt{13}/2 = 1.8$

acceleration: $a(t) = \langle -\cos t, -2\sin t\rangle$, with $a(\pi/6) = \langle -\frac{\sqrt{3}}{2}, -1\rangle$



3. (20 points) Use chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = e^y \cos x \sin x + \ln(3x + y^2)$, with $x = s^2 t$ and $y = \frac{s}{t^2}$. You may leave your answers in terms of x, y, s and t.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(e^y(-\sin^2 x + \cos^2 x) + \frac{3}{3x + y^2}\right) \cdot 2st + \left(e^y \cos x \sin x + \frac{2y}{3x + y^2}\right) \cdot \frac{1}{t^2}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(e^y(-\sin^2 x + \cos^2 x) + \frac{3}{3x + y^2}\right) \cdot s^2 + \left(e^y \cos x \sin x + \frac{2y}{3x + y^2}\right) \cdot \left(\frac{-2s}{t^3}\right)$$

4. (20 points) Find the directional derivative of $f(x, y, z) = \frac{x^2+3y-e^{2z}}{7y+\pi}$ at the point $(1, 1, 0)$ in the direction toward the point $(1, 0, 1)$.

Solution: The unit vector in the direction of $(1, 0, 1)$ is $v = \langle 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. And so

$$\begin{aligned}
D_v(1,1,0) &= \frac{\partial f}{\partial x}(1,1,0)(0) + \frac{\partial f}{\partial y}(1,1,0)\left(\frac{-1}{\sqrt{2}}\right) + \frac{\partial f}{\partial z}(1,1,0)\left(\frac{1}{\sqrt{2}}\right) \\
&= 0 + \frac{3(7y+\pi) - 7(x^2 + 3y - e^{2z})}{(7y+\pi)^2}(1,1,0)\left(\frac{-1}{\sqrt{2}}\right) + \frac{-2e^{2z}}{7y+\pi}(1,1,0)\left(\frac{1}{\sqrt{2}}\right) \\
&= \frac{3\pi}{(7+\pi)^2}\frac{-1}{\sqrt{2}} + \frac{-2}{(7+\pi)}\frac{1}{\sqrt{2}} = \frac{-3\pi - 2(7+\pi)}{\sqrt{2}(7+\pi)^2} = -\frac{14 + 5\pi}{\sqrt{2}(7+\pi)^2}
\end{aligned}$$

5. (20 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 8x - 4z$ subject to the constraint $x^2 + 10y^2 + z^2 = 5$.

Solution: Let $f(x, y, z) = 8x - 4z$ and $g(x, y, z) = x^2 + 10y^2 + z^2 = 5$. We solve $\nabla f = \lambda \nabla g$:

$$8 = 2x\lambda \quad \Rightarrow \quad x = \frac{4}{\lambda}$$

$$0 = 20y\lambda \quad \Rightarrow \quad y = 0$$

$$-4 = 2z\lambda \quad \Rightarrow \quad z = \frac{-2}{\lambda}$$

Replacing x, y and z in the constraint equation, we have: $\dfrac{16}{\lambda^2} + 0 + \dfrac{4}{\lambda^2} = 5 \Rightarrow \lambda = \pm 2$.

If $\lambda = 2$, we have $f(2, 0, -1) = 20 \Rightarrow f$ has its max of 20 at $(2, 0, -1)$
If $\lambda = -2$, we have $f(-2, 0, 1) = -20 \Rightarrow f$ has its min of -20 at $(-2, 0, 1)$